Taylor Theorem

Assume f is differentiable $k+1$ times over (a,b). Let $f^{(i)}$ be the i-th derivative function, assume $f^{(i)}$ is continuous for all $i = 0, 1, \ldots, k$.

Then there exists $c \in (x_0, x)$ s.t.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} \cdot (x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!} \cdot (x - x_0)^k + \frac{f^{(k+1)}(c)}{(k+1)!}$$

1. Remark: base case $k = 0$

There exists $c \in (x_0, x)$ s.t. $f(x) = f(x_0) + f'(c)(x - x_0)$, i.e. there exists $c \in (x_0, x)$ s.t. $f'(c) = \frac{f(x)-f(x_0)}{x-x_0}$. This is exactly the MVT.

2. Proof:

Apply MVT once. Recall MVT (3), f, g satisfy the usual condition $[a,b] \to \mathbb{R}$. Then there exists $c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

In this case we want $\star : \frac{f(u)-f(v)-f'(v)(u-v)}{\frac{(u-v)^2}{2}} = f''(c)$. Choose $F(x) = f(u) - f(x) - f'(x) \cdot (u - x)$. Then $F(v) =$ numerator of LHS $\cdot f(\star)$ and $F(u) = 0$ and $G(v) =$ denominator of LHS of $(\star)$ and $G(u) = 0$.

Then $\frac{F(u)-F(v)}{G(u)-G(v)} =$ LHS of $(\star)$. Now MVT implies $\frac{F'(c)}{G'(c)}$. Now $F'(x) = (f(u) - f(x) - f'(x) \cdot (u-x))' = -f'(x) - f''(x)(u-x) + f'(x) = -f''(x)(u-x)$. Thus, $G'(x) = (u-x) \cdot (-1) = -(u-x)$. We plug back in yielding $\frac{F'(c)}{G'(c)} = \frac{-f''(c)(u-c)}{-(u-c)} = f''(c)$.

3. Remark: for all k, do the same but with:

$$F(x) = f(u) - f(x) - f'(x) \cdot (u - x) - \cdots - \frac{f^{(k)}(x)}{k!}(u-x)^k \implies G(x) = \frac{(u-x)^{k+1}}{(k+1)!}$$